CHINESE JOURNAL OF PHYSICS VOL. 38, NO. 4 AUGUST 2000

Spherical Harmonics and Monopole Harmonics

M. K. Fung
Department of Physics, National Taiwan Normal University,
Taipei, Taiwan 116, R.O.C.
(Received November 18, 1999)

We present here a new method for constructing the spherical harmonics wavefunctions. We introduce two chiral states u' and v' of chirality $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Products of $u'v'$ of zero chirality, projected onto the angular momentum states, yield the spherical harmonics. We use similar techniques to construct the monopole harmonics.

PACS. 03.65.–w – Quantum mechanics.
PACS. 14.80.Hv – Magnetic monopoles.

I. Introduction

Spherical harmonics are quantum mechanical wavefunctions for the angular momentum operators. Needless to say they are important and well-studied. The conventional way of study is through the method of separation of variables. The angular part of the Laplacian, the hamiltonian for the free particle, is studied as an eigenvalue problem. In fact, the use of the algebraic method of raising and lowering operators obviates the need for the explicit form of the eigenfunctions.

An ingenious method [1] was introduced by Schwinger to study the representations of the $SU(2)$ group. The angular momentum operators are represented by two sets of boson-oscillator operators. Many interesting results of the angular momentum theory can be deduced algebraically without any explicit form of the boson-oscillator operators. Bargmann [2] replaced the Hilbert space by a function space. That is, each boson raising operator is represented by a complex variable. Thus analytical methods can be implemented.

In the approach of Bargmann the complex variables u and v are definite eigenfunctions for the magnetic quantum number. We can rotate these states to get two chiral states u' and v' of chirality $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. There is an ambiguity in the definition of the chiral states related to the Hopf fibration [3]. This ambiguity does not happen for the theory of orbital angular momentum because of the constraint that states for the orbital angular momentum are of zero chirality. Products of $u'v'$ will satisfy this constraint. We find that these states projected along the angular momentum states will yield the spherical harmonics.

This approach can be applied to the problem of the monopole harmonics [4]. For the case of a particle in a monopole field the angular momentum gets an extra contribution from the monopole field. Its chirality is $-eg$, where e and g are the electric and magnetic pole strength, respectively. Hence suitable products of u' and v' will satisfy the constraint for nonzero chirality. The monopole harmonics can be easily written down. The ambiguity due to the Hopf fibration mentioned above corresponds to the degree of freedom of the gauge field, a $U(1)$ group.

The paper is organised as follows. In section 2 we discuss in details how the chiral states u' and v' are introduced. The preliminary works of Schwinger and Bargmann are reviewed. The spherical harmonics are worked out. In section 3 we study the problem of a particle in a monopole field, emphasising the role of the extra contribution of the magnetic field to the angular momentum. We work out the monopole harmonics in section 4 following our method of chiral states. In the final section we make our conclusion.

II. Spherical harmonics a là schwinger and bargmann

We study here spherical harmonic functions via Schwinger's method of oscillator operators. In the latter part of this section we implement Bargmann's method of representing the wavefunction by two holomorphic functions denoting the spinor co-ordinates. A rotated wavefunction is used to render the wavefunction in the co-ordinate representation. To begin, we start with the free field Lagrangian

$$L = \frac{1}{2} m \dot{\mathbf{r}}^2 \,, \tag{1}$$

where m is the mass of the particle and $\dot{\mathbf{r}}$ is the velocity vector of the free particle in three dimensions. The hamiltonian is

$$H = \frac{\mathbf{p}^2}{2m} \,, \tag{2}$$

where $\mathbf{p}$ is the conjugate momentum vector and is proportional to the velocity vector as

$$\mathbf{p} = m\dot{\mathbf{r}} \,. \tag{3}$$

The co-ordinates and the conjugate momenta obey the canonical commutation relations

$$\left[r^i, r^j\right] = \left[p^i, p^j\right] = 0 \,, \tag{4}$$

$$\left[r^i, p^j\right] = i\hbar\delta^{ij} \,. \tag{5}$$

The hamiltonian can be decomposed into a radial part as well as a transverse part

$$H = \frac{p_r^2}{2m} + \frac{1}{2mr^2} \left(\mathbf{r} \times \mathbf{p}\right)^2 \,. \tag{6}$$

So we can define the angular momentum vector as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \,, \tag{7}$$

where the commutation relations of the various components of $\mathbf{L}$ are given as a consequence of the canonical commutation relations by

$$\left[L^i, L^j\right] = i\epsilon_{ijk} L^k \,. \tag{8}$$

The angular part of the full hamiltonian can be studied separate from the radial part. Hence in the following we shall study the angular momentum hamiltonian denoted by

$$H_{ang} = \mathbf{L}^2 \,. \tag{9}$$

This hamiltonian will give rise to the spherical harmonic wavefunction.

It takes the genius of Schwinger [1] to represent the angular momentum operators in terms of two set of boson-oscillator operators. Schwinger introduced spin creation and annihilation operators associated with a given spatial reference system, $a^+_\zeta = (a^+_+, a^+_-)$ and $a_\zeta = (a_+, a_-)$, which satisfy

$$\left[a_\zeta, a_{\zeta'}\right] = 0\,, \qquad \left[a^+_\zeta, a^+_{\zeta'}\right] = 0\,, \tag{10}$$

$$\left[a_\zeta, a^+_{\zeta'}\right] = \delta_{\zeta\zeta'}\,. \tag{11}$$

The number of spins and the resultant angular momentum are then given by

$$n = \sum_\zeta a^+_\zeta a_\zeta = n_+ + n_-\,, \tag{12}$$

$$\mathbf{L} = \sum_{\zeta,\zeta'} a^+_\zeta \langle\zeta|\frac{1}{2}\boldsymbol{\sigma}|\zeta'\rangle a_{\zeta'}\,, \tag{13}$$

where σ are the hermitian Pauli spin matrices. The components of $\mathbf{L}$ appear as

$$L_+ = L_1 + iL_2 = a^+_+ a_-\,, \tag{14}$$

$$L_- = L_1 - iL_2 = a^+_- a_+\,, \tag{15}$$

$$L_3 = \frac{1}{2}(a^+_+ a_+ - a^+_- a_-) = \frac{1}{2}(n_+ - n_-)\,. \tag{16}$$

The square of the angular momentum can be written as

$$\mathbf{L}^2 = l(l+1)\,, \tag{17}$$

where the quantum number l is given by

$$l = \frac{1}{2}n = \frac{1}{2}(a^+_+ a_+ + a^+_- a_-)\,. \tag{18}$$

Accordingly, a state with a definite magnetic quantum number m

$$m = \frac{1}{2}(n_+ - n_-), \qquad l = \frac{1}{2}(n_+ + n_-)\,, \tag{19}$$

is defined to be

$$|l, m\rangle = \frac{(a^+_+)^{l+m}(a^+_-)^{l-m}}{\sqrt{(l+m)!}\sqrt{(l-m)!}}\Psi_0\,, \tag{20}$$

$$a_{\pm}\Psi_0 = 0\,. \tag{21}$$

This is an operator approach and many useful results can be obtained without explicitly finding the wavefunctions. However, in order to write down wavefunctions we follow the method of Bargmann [2], introducing two holomorphic variables u and v. The variables u and v denote the eigenfunction with $L_3 = \pm\frac{1}{2}$,

$$u \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{22}$$

We can then identify

$$a_+ \to \partial_u\,, \quad a_- \to \partial_v\,, \quad a_+^+ \to u\,, \quad a_-^+ \to v\,. \tag{23}$$

In this representation the angular momentum operators are represented as

$$L_+ = u\frac{\partial}{\partial v}\,, \qquad L_- = v\frac{\partial}{\partial u}\,, \tag{24}$$

$$L_3 = \frac{1}{2}(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}), \tag{25}$$

and the total angular momentum operator is

$$l = \frac{1}{2}n = \frac{1}{2}(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v})\,. \tag{26}$$

The inner product for holomorphic functions of u is defined as

$$(f, f') = \int \overline{f(u)} f'(u)[du]\,, \tag{27}$$

where the measure $[du]$ is defined to be

$$[du] = \frac{ie^{-\bar{u}u}}{2\pi}\,. \tag{28}$$

It follows that

$$(u^h, u^{h'}) = h!\delta_{hh'}\,. \tag{29}$$

Similar equations hold for holomorphic functions of v.

Now the ground state Ψ_0 is unity and the angular eigenstates are given by

$$|l, m\rangle = \frac{u^{l+m}v^{l-m}}{\sqrt{(l+m)!}\sqrt{(l-m)!}}\,. \tag{30}$$

We now consider the effect of a rotation on the system.The spinor co-ordinates are rotated by an $SU(2)$ matrix

$$U = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \tag{31}$$

with

$$\bar{\alpha}\alpha + \bar{\beta}\beta = 1\,. \tag{32}$$

The spinor co-ordinates u and v are transformed as

$$\tilde{u} = \alpha u + \beta v\,, \tag{33}$$

$$\tilde{v} = -\bar{\beta} u + \bar{\alpha} v\,. \tag{34}$$

The $SU(2)$ matrix is related to the spatial rotation by the formula

$$U = \exp(-i\frac{\gamma}{2}\mathbf{n}\cdot\boldsymbol{\sigma})\,, \tag{35}$$

where $\mathbf{n}$ is the axis of rotation with magnitude γ. Now we can utilise the rotation

$$U(\theta,\phi) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2}e^{i\phi} & \cos\frac{\theta}{2} \end{pmatrix}, \tag{36}$$

to rotate the z-axis to the $\hat{\mathbf{r}}$ direction where

$$\hat{\mathbf{r}} = (\sin\theta\cos\phi\,, \sin\theta\sin\phi\,, \cos\theta)\,. \tag{37}$$

The transformed states

$$u' = \alpha u + \beta v\,, \tag{38}$$

$$v' = -\bar{\beta} u + \bar{\alpha} v\,, \tag{39}$$

where

$$\alpha = \cos\frac{\theta}{2}\,, \tag{40}$$

$$\beta = \sin\frac{\theta}{2}e^{i\phi}\,, \tag{41}$$

are eigenstates of $\hat{\mathbf{r}}\cdot\mathbf{L}$ such that

$$(\hat{\mathbf{r}}\cdot\mathbf{L})u' = \frac{1}{2}u'\,, \qquad (\hat{\mathbf{r}}\cdot\mathbf{L})v' = -\frac{1}{2}v'\,. \tag{42}$$

In fact we should make the identification $(\alpha e^{i\delta}, \beta e^{i\delta}) \sim (\alpha, \beta)$. This is the famous Hopf mapping [3] of $S^3 \mapsto S^2$. We defer the discussion of this degree of freedom later. This degree of freedom does not matter for ordinary orbital angular momentum. In this sense we can say that v' is the diametrically opposite eigenstate with θ and ϕ increased by an angle of π.

We are now going to discuss how to get the spherical harmonics for the spatial orbital angular momentum. The following remark is useful. For orbital angular momentum we have the constraint

$$\hat{\mathbf{r}} \cdot \mathbf{L} = 0\,. \tag{43}$$

So it is natural to define

$$|\theta\,,\phi\rangle = \frac{(u')^l (v')^l}{l!}\,, \tag{44}$$

$$= \frac{(\alpha u + \beta v)^l (-\bar{\beta} u + \bar{\alpha} v)^l}{l!}\,, \tag{45}$$

and the constraint equation (43) is satisfied automatically. The spherical harmonics are then represented by

$$\langle \theta\,,\phi | l\,,m\rangle = \int \frac{(\bar{u}')^l (\bar{v}')^l}{l!} \frac{(u)^{l+m} (v)^{l-m}}{\sqrt{(l+m)!}\sqrt{(l-m)!}} [du][dv]\,. \tag{46}$$

Using the above formula it is easy to work out the spherical harmonic wavefunctions in the co-ordinate representation. We list here a few of the spherical harmonics :

$$\begin{aligned}
\langle \theta\,,\phi | 0\,,0\rangle &= 1\,, \\
\langle \theta\,,\phi | 1\,,\pm 1\rangle &= \mp\sqrt{\tfrac{1}{2}} \sin\theta e^{\pm i\phi}\,, \\
\langle \theta\,,\phi | 1\,,0\rangle &= \cos\theta\,, \\
\langle \theta\,,\phi | 2\,,\pm 2\rangle &= \sqrt{\tfrac{3}{8}} \sin^2\theta e^{\pm i2\phi}\,, \\
\langle \theta\,,\phi | 2\,,\pm 1\rangle &= \mp\sqrt{\tfrac{3}{2}} \sin\theta\cos\theta e^{\pm i\phi}\,, \\
\langle \theta\,,\phi | 2\,,0\rangle &= \tfrac{1}{2}(3\cos^2\theta - 1)\,.
\end{aligned}$$

It is understood that the measure for integrating the angular variables is $(2l+1)\sin\theta d\theta d\phi/4\pi$.

The angular momentum operators operating on the spherical harmonics are easily verified to be

$$L_3 = -i\frac{\partial}{\partial\phi}\,, \tag{47}$$

$$L_\pm = e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\,. \tag{48}$$

III. Dynamics of an electron in the field of a monopole

We consider the motion of an electron of charge e and mass m in the field of a monopole g located at the origin. The equation of motion is given by the Lorentz force law,

$$m\ddot{\mathbf{r}} = \frac{eg}{r^3}(\dot{\mathbf{r}} \times \mathbf{r})\,. \tag{49}$$

The corresponding Lagrangian is given by

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 + e\dot{\mathbf{r}} \cdot \mathbf{A}\,, \tag{50}$$

where $\mathbf{A}$ is the vector potential for the monopole field,

$$\nabla \times \mathbf{A} = \mathbf{B} = \frac{g\mathbf{r}}{r^3}\,, \tag{51}$$

and the conjugate momenta $\mathbf{p}$ are

$$\mathbf{p} = m\dot{\mathbf{r}} + e\mathbf{A}\,. \tag{52}$$

In our approach we employ the noncanonical equal-time commutation relations

$$\left[r^i, r^j\right] = 0\,, \tag{53}$$

$$\left[r^i, \dot{r}^j\right] = \frac{i\hbar}{m}\delta^{ij}\,, \tag{54}$$

$$\left[\dot{r}^i, \dot{r}^j\right] = \frac{i\hbar e}{m^2}\epsilon^{ijk}B^k\,. \tag{55}$$

Now these commutation relations do not involve gauge-variant quantities.

This problem is rotationally invariant and the symmetry generators for the rotational group are the angular momentum operators,

$$\mathbf{J} = m\mathbf{r} \times \dot{\mathbf{r}} - \frac{eg\mathbf{r}}{r}\,. \tag{56}$$

The angular momentum operators consist of a mechanical part as well as a field part. It is easy to verify that these angular momentum operators satisfy the usual commutation relations

$$\left[J^i, J^j\right] = i\epsilon^{ijk}J^k\,. \tag{57}$$

It is also easy to prove that these angular momentum operators are constants of motion.

The problem of an electron in a monopole field has been completely solved, both classically and quantum mechanically. Jackiw [5] has uncovered that this problem has additionally hidden symmetry, leading to the exact solvability of the problem. The complete symmetry is $O(2,1) \times O(3)$. We shall be more interested in the rotational symmetry of the problem. Indeed, Wu and Yang [4] studied the monopole harmonics in the sense as a section of the monopole line bundle.

Much earlier, Tamm [6] studied the Schrödinger equation of an electron in a monopole field. In the next section we shall exploit the method of chiral states to work out the monopole harmonics.

IV. Monopole harmonics

Explicitly one can easily verify

$$\mathbf{J}^2 = [m\mathbf{r} \times \dot{\mathbf{r}}]^2 + (eg)^2 \,. \tag{58}$$

The full hamiltonian can be decomposed into a radial as well as an angular part,

$$H = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}m\frac{[\mathbf{r} \times \dot{\mathbf{r}}]^2}{r^2} \,. \tag{59}$$

The radial and the angular parts are separable. Henceforth we shall study the angular hamiltonian,

$$H_{ang} = \mathbf{L}^2 \,, \tag{60}$$

where $\mathbf{L}$ is the mechanical angular momentum defined by the velocities

$$\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}} \,. \tag{61}$$

In this case the velocities are not proportional to the linear momentum. The true angular momentum operators generating rotations are the J_i's and the operators $\mathbf{r}$, $\dot{\mathbf{r}}$ and $\mathbf{L}$ are vectors under $\mathbf{J}$. We can use equation (58) to rewrite the hamiltonian for the angular part as

$$H_{ang} = \mathbf{J}^2 - q^2 \,, \tag{62}$$

where we have define

$$q = eg \,, \tag{63}$$

which is a half integer or integer [7]. We can represent the angular momentum operators by the spinor variables u and v as

$$J_+ = u\frac{\partial}{\partial v} \,, \qquad J_- = v\frac{\partial}{\partial u} \,, \tag{64}$$

$$J_3 = \frac{1}{2}(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}) \,. \tag{65}$$

In the field of the monopole the projection of $\mathbf{J}$ along $\hat{\mathbf{r}}$ is not zero but equals $-q$,

$$\hat{\mathbf{r}} \cdot \mathbf{J} = -q \,. \tag{66}$$

The sign of q matters and in the following discussion we shall take q negative. As in the previous case, the rotated state

$$u' = e^{i\chi} \left(\alpha u + \beta v\right) \,, \tag{67}$$

is an eigenstate of $\hat{\mathbf{r}} \cdot \mathbf{J}$ with eigenvalue $+\frac{1}{2}$ while the state

$$v' = e^{-i\chi}\left(-\bar{\beta}u + \bar{\alpha}v\right) , \tag{68}$$

is an eigenstate of $\hat{\mathbf{r}} \cdot \mathbf{J}$ with eigenvalue $-\frac{1}{2}$. Unlike the previous case we can introduce a phase $e^{i\chi}$ which represents the gauge degree of freedom. We can choose suitable gauge to render the monopole harmonics in a special form. Now we can propose the state

$$|q, n, \theta, \phi\rangle = e^{i2|q|\chi} \frac{(u')^{2|q|}(u'v')^n}{\sqrt{n!}\sqrt{(n+2|q|)!}} , \tag{69}$$

as the eigenstate of $\hat{\mathbf{r}} \cdot \mathbf{J}$ of eigenvalue $-q$. The quantum number n running from 0, 1, 2, ... labels the n^{th} Landau energy level as the monopole harmonic problem is just the Landau problem on a sphere [8]. So

$$j = |q| + n , \tag{70}$$

where j is the quantum number for the total angular momentum. The energy levels are given by

$$E = j(j+1) - q^2 , \tag{71}$$

with degeneracy $2j + 1$ or $2|q| + 2n + 1$.

We shall take $\chi = 0$,

$$e^{i\chi} = 1 , \tag{72}$$

so that the operator J_3 will take the form $-i\partial/\partial\phi - q$ in this gauge. We shall call this the holomorphic gauge. In this gauge we can take limits to get the Landau problem in the symmetric gauge [9]. The monopole harmonics can be obtained by projecting Eq. (69) onto the angular momentum states $|j, m\rangle$. We list here a few of the monopole harmonics for $q = -\frac{1}{2}$,

$$\langle\theta, \phi, -\tfrac{1}{2}, 0\ |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \cos\tfrac{\theta}{2} , \tag{73}$$

$$\langle\theta, \phi, -\tfrac{1}{2}, 0\ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = e^{-i\phi}\sin\tfrac{\theta}{2} , \tag{74}$$

$$\langle\theta, \phi, -\tfrac{1}{2}, 1\ |\tfrac{3}{2}, \tfrac{3}{2}\rangle = -e^{i\phi}\sqrt{3}\cos^2\tfrac{\theta}{2}\sin\tfrac{\theta}{2} , \tag{75}$$

$$\langle\theta, \phi, -\tfrac{1}{2}, 1\ |\tfrac{3}{2}, \tfrac{1}{2}\rangle = \cos\tfrac{\theta}{2}\left(\cos^2\tfrac{\theta}{2} - 2\sin^2\tfrac{\theta}{2}\right) , \tag{76}$$

$$\langle\theta, \phi, -\tfrac{1}{2}, 1\ |\tfrac{3}{2}, -\tfrac{1}{2}\rangle = e^{-i\phi}\sin\tfrac{\theta}{2}\left(2\cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2}\right) , \tag{77}$$

$$\langle\theta, \phi, -\tfrac{1}{2}, 1\ |\tfrac{3}{2}, -\tfrac{3}{2}\rangle = e^{-i2\phi}\sqrt{3}\sin^2\tfrac{\theta}{2}\cos\tfrac{\theta}{2} , \tag{78}$$

and for $q = -1$,

$$\langle\theta, \phi, -1, 0\ |1, 1\rangle = \cos^2\tfrac{\theta}{2} \tag{79}$$

$$\langle\theta, \phi, -1, 0\ |1, 0\rangle = e^{-i\phi}\tfrac{2}{\sqrt{2}}\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}, \tag{80}$$

$$\langle\theta, \phi, -1, 0\ |1, -1\rangle = e^{-2i\phi}\sin^2\tfrac{\theta}{2} , \tag{81}$$

It is understood that the measure for integrating the angular variables is $(2j+1)\sin\theta d\theta d\phi/4\pi$.

The angular momentum operators in this gauge are found to be

$$J_3 = -i\frac{\partial}{\partial\phi} - q\,, \tag{82}$$

$$J_\pm = e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi} - q\tan\frac{\theta}{2}\right)\,. \tag{83}$$

V. Conclusion

In this paper we do not offer any new results for the angular momentum theory. Rather, we give a new method of constructing the spherical harmonics. This can be called the method of synthesis. The basic units are the chiral states u' and v'. The constraint equation for the chirality is the key factor of the calculation. We find that the monopole harmonics can be constructed in this unified view. The gauge degree of freedom appears quite naturally. We are sure that this same line of thought can be applied to the problem of the $SU(2)$ monopole harmonics [10] as this case is also a Hopf fibration. It is interesting to study whether this method can be extended to study representations of group outside the scope of Hopf mapping.

Acknowledgment

This work is supported in part by the National Science Council of R.O.C. under the contract NSC-89-2112-M-003-002.

References

[1] J. Schwinger, *On angular momentum*, (Atomic Commission, NYO-3071, unpublished, 1952).
[2] V. Bargmann, Rev. Mod. Phys. **34**, 829 (1962).
[3] H. Hopf, Math. Ann. **104**, 637 (1931).
[4] T. T. Wu and Chen Ning Yang, Nucl. Phys. B**107**, 365 (1976).
[5] R. Jackiw, Ann. Phys. (N.Y.) **129**, 201 (1980).
[6] Ig. Tamm, Z. Phys. **71**, 141 (1931).
[7] P.A.M. Dirac, Proc. Roy. Soc. A**133**, 60 (1931).
[8] F.D.M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[9] M. K. Fung and Y. F. Wang, Chin. J. Phys. **37**, 435 (1999).
[10] C. N. Yang, J. Math. Phys. **19**, 2622 (1978).